Homework 7 Solution

1. Sec. 5.2 Q11

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

$\exists$ invertible Q s.t. $Q^{-1}AQ = R$ is an upper triangular matrix

$$f_A(t) = \det(A - tI_n) = \det(R - tI_n) = (R_{11}-t)\cdots(R_{nn}-t)$$

since $\lambda_1 \cdots \lambda_k$ are distinct eigenvalues of A.

Therefore $R_{ii} \in \{\lambda_1 \cdots \lambda_k\}$ for $i = 1 \cdots n$

suppose $R_{ii} = \lambda_j$ occurs d_i times in $f_A(t)$

Since $(\lambda_j - t)^{m_j} \mid f_A(t)$ so $m_j \le d_j$

$n = \sum_{j=1}^{k} m_j \le \sum_{j=1}^{k} d_j = n$ so $m_j = d_j$ for $j = 1 \cdots k$.

Thus the diagonal entries of A are $\lambda_1 \cdots \lambda_k$

And each λ_i occurs m_i times.

(a) Note that $\operatorname{tr}(BC) = \operatorname{tr}(CB)$ $\forall$ $B, C \in M_{n\times n}$

$$\operatorname{tr}(A) = \operatorname{tr}(Q \cdot R Q^{-1}) = \operatorname{tr}(RQ^{-1} \cdot Q)$$

$$= \operatorname{tr}(R) = \sum_{1}^{n} R_{ii} = \sum_{1}^{k} m_i \lambda_i$$

(b) $\det(A) = f_A(0) = \prod_{1}^{n} R_{ii} = \prod_{1}^{k} \lambda_i^{m_i}$

2. Sec. 5.2 Q13

13. Let $A \in \mathsf{M}_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

(a) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad E_0 = \operatorname{span}(\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\})$

$A^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad E_0' = \operatorname{span}(\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}) \neq E_0$

(b) $(A - \lambda I_n)^t = A^t - \lambda I_n^t = A^t - \lambda I_n$

$$\begin{aligned}
\dim(E_\lambda) &= \dim(N(A-\lambda I_n)) \\
&= n - \dim(R(A-\lambda I_n)) \\
&= n - \operatorname{rank}(A-\lambda I_n) \\
&= n - \operatorname{rank}((A-\lambda I_n)^t) \\
&= n - \operatorname{rank}(A^t-\lambda I_n) \\
&= n - \dim(R(A^t-\lambda I_n)) \\
&= \dim(N(A^t-\lambda I_n)) \\
&= \dim(E_\lambda')
\end{aligned}$$

(c) A is diagonalizable $\Leftrightarrow$ Algebraic multiplicity of λ equals to geometric multiplicity of λ for all λ of A.

i.e. $\mu_A(\lambda) = \gamma_A(\lambda)$

$\mu_{A^t}(\lambda) = \mu_A(\lambda) = \gamma_A(\lambda) = \gamma_{A^t}(\lambda)$ for each eigenvalue λ of A^t

Thus A^t is diagonalizable.

3. Sec. 5.2 Q22

22. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\operatorname{span}(\{x \in \mathsf{V} : x \text{ is an eigenvector of } \mathsf{T}\}) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

① $\forall\ v \in E_{\lambda_i} \cap \left(\sum_{j \neq i} E_{\lambda_j}\right)$

$T(v) = \lambda_i \cdot v$ and $v = \sum_{j\neq i} v_j$ where $v_j \in E_j$ for $j \neq i$

$T(v) = \sum_{j \neq i} T(v_j) = \sum_{j\neq i} \lambda_j v_j$

$0 = T(v) - T(v) = \lambda_i \cdot \sum_{j\neq i} v_j - \sum_{j\neq i} \lambda_j v_j = \sum_{j\neq i} \underbrace{(\lambda_i - \lambda_j) v_j}_{\in E_{\lambda_j}}$

Thus $(\lambda_i - \lambda_j) v_j = 0$ which implies $v_j = 0$

Thus $v = 0$

Therefore $\sum_{j=1}^{k} E_{\lambda_j} = \bigoplus_{j=1}^{k} E_{\lambda_j}$

② $\because$ $E_{\lambda_j} \subset \operatorname{span}(\{x \in V : x \text{ is an eigen vector of } T\})$

$\therefore$ $E_{\lambda_1} + \cdots + E_{\lambda_k} \subset \operatorname{span}(\{x \in V : x \text{ is an eigen vector of } T\})$

③ $\forall\ v \in \operatorname{span}(\{x \in V : x \text{ is an eigen vector of } T\})$

$v = \underbrace{v_1 + v_2 + v_3}_{\lambda_1} + \underbrace{v_4 + v_5}_{\lambda_2} + \cdots + v_p$

group these v_j's based on eigen values.

$= \underset{\uparrow\ E_{\lambda_1}}{w_1} + \underset{\uparrow\ E_{\lambda_2}}{w_2} + \cdots + \underset{\uparrow\ E_{\lambda_k}}{w_k}$

$\in E_{\lambda_1} + \cdots + E_{\lambda_k}$

4. Sec. 5.4 Q13

13. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in$ V, prove that $w \in$ W if and only if there exists a polynomial $g(t)$ such that $w = g(\mathsf{T})(v)$.

$W = \text{span}(\{v, T(v), T^2(v), \cdots\})$ $v \neq 0$

($\Rightarrow$) $w \in W$, then $\exists\ a_0 \cdots a_k \in F$ s.t.

$w = a_0 v + a_1 T(v) + \cdots + a_k T^k(v)$

$= g(T)(v)$

where $g(t) = a_0 + a_1 t + \cdots + a_k t^k$

($\Leftarrow$) Since W is T-invariant. by exercise 4

W is $g(T)$-invariant.

$v \in W$. then $w = g(T)(v) \in W$

5. Sec. 5.4 Q23

23. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in$ W for all i. *Hint:* Use mathematical induction on k.

W is T-invariant, then $T(W) \subset W$

$T(v_j) = \lambda_j \cdot v_j$, $j = 1, \cdots, k$ and $\lambda_i \neq \lambda_j$ if $i \neq j$

- For $k=1$, if $v_1 \in W$, then $v_1 \in W$.
- Suppose this is true for case k
- If $v_1 + \cdots + v_{k+1} \in W$

then $\lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1} = T(v_1 + \cdots + v_{k+1}) \in W$

$(\lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1}) - \lambda_{k+1}(v_1 + \cdots + v_{k+1}) \in W$

i.e. $(\lambda_1 - \lambda_{k+1}) v_1 + \cdots + (\lambda_k - \lambda_{k+1}) v_k \in W$

Since $\lambda_j - \lambda_{k+1} \neq 0$ for $j = 1 \cdots k$.

$(\lambda_j - \lambda_{k+1}) v_j$ is eigen vector of T corresponding to λ_j.

By assumption, $(\lambda_j - \lambda_{k+1}) v_j \in W$, $j = 1, \cdots, k$.

Since $\lambda_j - \lambda_{k+1} \neq 0$, $v_j \in W$ for $j = 1, \cdots, k$

$v_{k+1} = (v_1 + \cdots + v_{k+1}) - v_1 - \cdots - v_k \in W$

Therefore $v_i \in W$ for $i = 1, \cdots, k+1$